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Dimension of discrete fractal spaces

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Abstract. A new definition of fractal dimension in the case of discrete metric spaces is given. It defines the dimension of an arbitrary unbounded subset X of the ν-dimensional lattice Z^ν.

1. Introduction

Recently the physics community working on applications of fractal geometry became aware [1-3] of slight differences in the definitions of fractal dimension that are being used. Apparently, if the fractal objects have sufficient scaling symmetry then it does not matter too much how the fractal dimension is calculated. But in many situations the scaling symmetry is absent and a careful choice of algorithm should be made.

In a strict sense any self-similar fractal [4] has perfect scaling properties. This means that there exists a transformation of the fractal object which rescales all distances by a constant factor $\lambda \neq 1$. A broader class of fractals have more complicated scaling properties. For example, the similarity transformation may rescale the object with different scale factors in different directions of n-dimensional space. It is in the latter case that explicit examples are cited in the literature [1-3] of fractal dimensions which depend on the definition one uses.

For many fractals the scaling properties emerge only asymptotically, either in the microscopic limit (as, e.g., in the Hénon attractor) or in the limit of large volumes. The so-called fractal lattices [5] are an example of the latter case. Sometimes the names *inner* and *outer* (or *local* and *global*) dimension are used to distinguish between fractal dimensions calculated in the two asymptotical regimes. The only definitions which explicitly give the inner fractal dimension are those of the Hausdorff dimension and of capacity. Other definitions implicitly assume full scaling behaviour. In view of the difficulties with disagreeing definitions mentioned above it is desirable to formulate an explicit definition for the outer fractal dimension. Such a definition is given here, albeit in the restricted context of fractal lattices.

2. The mass and box-counting definitions

Consider a subset X of the ν-dimensional lattice Z^ν. The obvious way to determine its fractal dimension d_f is by the scaling relation

$$|X \cap V(l)| \sim l^{d_f}$$

stating that the number of points in the intersection of X with a ν-dimensional box of side l varies as a power of l. (The word 'box' is used with the meaning of (hyper-)cube; instead of cubes one could use spheres by an obvious change of metric.) This is known as the 'mass' definition of the fractal dimension [6]. Remark that it is implicitly understood that the box is placed in such a way that the intersection with X contains a maximal number of points. Indeed, the intersection of an arbitrary box with X is empty with probability one (at least if the fractal dimension of the set X is strictly less than that of the lattice).

Alternatively one can use the box-counting algorithm [4, p 196] which can be adapted to fractal lattices as follows. A large box $V(l)$ is partitioned into small boxes $V_i(s)$ of side s. One counts the number of these small boxes containing at least one point of X. Let $n(l, s)$ denote the latter number. Then the formula for the fractal dimension d_f is

$$d_f = -\lim_{s,l\to\infty} \frac{1}{\ln s} \ln\left(\frac{n(l, s)}{|X \cap V(l)|}\right). \tag{1}$$

How the limits of large s and l should be taken is not too clear. In practice one makes an appropriate choice of the large box $V(l)$ and looks for a domain of s values where the quantity $n(l, s)$ shows scaling behaviour. Because the number of points in the boxes $V_i(s)$ scales as s^{d_f} the number of squares $n(l, s)$ varies asymptotically as

$$n(l, s) \sim |X \cap V(l)| s^{-d_f}.$$

Using the latter expression one immediately verifies that the box-counting formula agrees with the mass definition if scaling arguments hold.

3. Definition

The new definition proposed in the present paper makes use of coverings of a finite part $X \cap V(l)$ of the set X with sets A_i whose diameter $\delta(A_i)$ is larger than or equal to a minimal value s. (The A_i may overlap—a partition is not required.) The covering is varied so as to obtain a minimal value for the expression

$$\sum_i \left(\frac{\delta(A_i)}{s}\right)^d.$$

The dimension d which appears is a free parameter, not related to the dimension ν of the host lattice. The obtained minimum is used to construct a function

$$m_d(X, s) = \lim_{l\to\infty} \frac{1}{|X \cap V(l)|} \min \sum_i \left(\frac{\delta(A_i)}{s}\right)^d. \tag{2}$$

The convergence of the limit is discussed in the appendix.

For small values of the dimension d one always finds $m_d(X, s) = 0$, independent of the value of s. In the limit $d \to \infty$ the function m_d involves the minimal number of boxes needed to cover $X \cap V(l)$. Hence m_∞ is the analogue of the capacity.

The following inequalities can be easily derived. First use one box A_i of diameter s for each point of the intersection $X \cap V(l)$. One obtains

$$0 \leq m_d(X, s) \leq 1.$$

Let $s \leqslant s'$. Each covering of $X \cap V(l)$ with elements of diameter $\delta(A_i) \geqslant s'$ is a covering with elements satisfying $\delta(A_i) \geqslant s$. There follows

$$s \leqslant s' \qquad \text{implies} \qquad m_d(X, s) \leqslant (s'/s)^d m_d(X, s'). \tag{3}$$

If $m_d(X, s)$ does not vanish for one value of s, then it is strictly positive for all $s' \geqslant s$. Depending on the value of d two possibilities are left: either $m_d(X, s)$ is identically zero as a function of s, or $m_d(X, s)$ is strictly positive for large enough values of s. It is also obvious that $m_d(X, s)$ is an increasing function of d, or is at most constant.

Now the fractal dimension d_f of the set X is defined as the value of d such that the function $m_d(X, s)$ is identically zero for smaller values and is non-zero for larger values of d. Remark that the inverse $1/m_d(X, s)$ is the d-dimensional density of points in X and is the analogue of the d-dimensional Hausdorff measure. If the set X is self-similar then the mass definition applies and is expected to agree with the new definition.

4. Examples

Let us consider three subsets X of the natural numbers, and determine their fractal dimension.

The number of primes smaller than a given integer N varies as $N/\ln N$. Hence the set X of all prime numbers has a vanishing density and its fractal dimension does not follow in a trivial way. But one verifies immediately that $m_d(X, s) = 0$ for all $d < 1$, and that $m_1(X, 1) = 1$. One concludes that the fractal dimension d_f of the set of prime numbers equals 1.

Fix a positive integer $x \geqslant 2$. Let X be the set of all xth powers of the natural numbers

$$X = \{1, 2^x, 3^x, \ldots\}.$$

A scaling argument immediately tells us that $d_f = 1/x$ (mass definition of the fractal dimension). Remark that the box-counting algorithm is not suited for this example. For special values of the size s of the boxes $(s = (p+1)^x - p^x + 1, p = 1, 2, \ldots)$ the number of non-empty boxes can be calculated analytically. It nowhere shows a power-law behaviour of the desired type. The new definition immediately gives $d_f \geqslant 1/x$. The argument to show equality is somewhat longer. Consider an interval A_i covering the points $p^x \ldots (p+s)^x$. It has length $\delta(A_i) = (p+s)^x - p^x + 1$ and covers $s+1$ points. One checks that $\delta(A_i)^{1/x} \geqslant s+1$. Hence the contribution of any term $\delta(A_i)$ in the definition of $m_d(X, s)$ with $d = 1/x$ is always larger than the number of points it covers. One concludes that $m_{1/x}(X, s) \geqslant 1$. Hence $d_f \leqslant 1/x$ follows.

Consider the following discrete analogue of the asymmetric Cantor set [7]. Sets C_n are constructed in an iterative way. Start with $C_0 = \{0\}$ and $C_1 = \{0, 1\}$ and construct C_2 by taking the union of C_1 and C_0, the latter shifted by two units:

$$C_2 = C_1 \cup \{C_0 + 2\} = \{0, 1, 2\}.$$

In the same way the set C_n is constructed by taking the union of C_{n-1} and C_{n-2}, the latter shifted in such a way that the largest element of C_n equals 2^{n-1}. The union of all sets C_n is the fractal object $C_\infty = \{0, 1, 2, 3, 4, 6, 7, 8, 12, 13, 14, 15, 16, 24, \ldots\}$. It has many properties in common with the asymmetric Cantor set of [7]. In particular the fractal dimension d_f according to the mass definition can be calculated and equals $\ln g/\ln 2$ where g is the golden ratio. It is the same as the Hausdorff dimension of the asymmetric Cantor set. (One immediately checks that the number of points in the set C_n equals the Fibonacci number F_{n+1} which asymptotically increases as g^n.) One verifies that $m_d(C_\infty, s) = s^{-d_f}$ for all $d > d_f$ and $m_d(C_\infty, s) = 0$ for all $d < d_f$. Hence the new definition and the mass definition coincide in this case.

5. Numerical determination

As a test of the practical usefulness of the function $m_d(X, s)$ a numerical determination of the dimension of the previous example C_∞ has been carried through. The approximant C_{10} was used. It contains 144 integers, the last of which is 512. The quantity

$$s^d m_d(s) = \frac{1}{|C_{10}|} \min \sum_i \delta(A_i)^d$$

was evaluated ($\{A_i\}_i$ is a covering of C_{10} with $\delta(A_i) \geq s$). See figure 1. Each curve in the figure takes about 1 min of computing time on a VAX 8200 computer. One observes that at $d = 0.694 \sim d_f$ the function $m_d(s)$ shows scaling behaviour in the range $s = 1$–512. At larger values of d ($d = 0.7$ and larger) there is a systematic deviation of scaling with exponent d. In this way a numerical determination of d_f with an accuracy of about 1% is achieved. At smaller values of d, $d < d_f$, one finds also that the function $m_d(s)$ scales as s^{-d}. This is an artefact due to the finite size of the sample. The coefficient in front of the power law tends to zero as the size of the sample is increased. (Indeed, $m_d(s)$ has to be zero for the infinite sample as long as $d < d_f$ holds.)

It would be interesting to apply the present definition to situations where one expects the different definitions to give different values for the fractal dimension. However, the minimisation which occurs in the definition of $m_d(s)$ poses numerical problems for two- and higher-dimensional host lattices. A first attempt to apply the definition on subsets of Z^2 is found in [8].

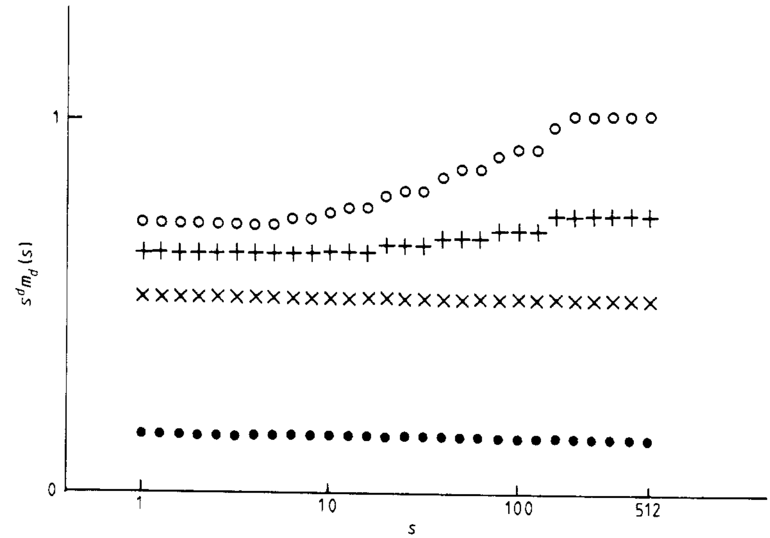

Figure 1. $s^d m_d(s)$ as a function of s for the finite sample $\{0, 1, \ldots, 512\}$ of the fractal lattice C_∞ for several values of d: 0.5 (●), 0.694 (×), 0.75 (+) and 0.8 (○).

6. Scaling

The determination of the dimension of fractal lattices can be simplified by the expected scaling behaviour of the function $m_d(X, s)$ for d values in the vicinity of, but slightly larger than d_f. Let us introduce the following function

$$D(d) = -\lim_{s\to\infty} \sup \ln m_d(X, s)/\ln s \qquad d > d_f. \tag{4}$$

(In numerical work one does not take the limit of large s, but for values of d close to d_f the scaling behaviour of $m_d(X, s)$ allows the determination of $D(d)$ as a slope in a log-log plot.)

One can show that the inequality $D(d) \leqslant d$ holds, and that the function $D(d)$ is decreasing or constant (see the appendix). Hence the typical situation is that of figure 2. If one assumes that $m_d(X, s)$ scales as s^{-d_f} for $d \to d_f$ then the inequality $D(d) \leqslant d$ becomes an equality in the limit $d = d_f$.

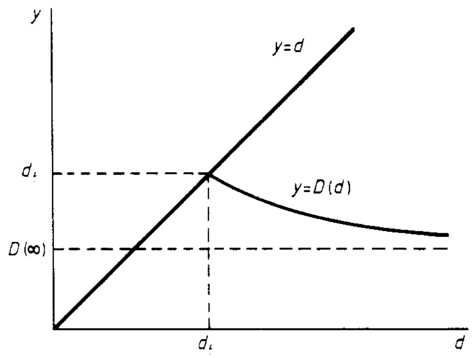

Figure 2. Expected behaviour of the function $D(d)$ as a function of $d > d_f$.

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Appendix

Consider a finite subset A of Z^ν and introduce the notation

$$m_d^A(s) = \frac{1}{|X \cap A|} \min \sum_i \left(\frac{\delta(A_i)}{s}\right)^d \tag{A1}$$

where the minimum is taken over all coverings $\{A_i\}_i$ of $X \cap A$ with sets A_i of diameter equal to or larger than s. The quantity m_d^A satisfies the following subadditivity property: let $A = A' \cup A''$, then one has

$$m_d^A(s) \leqslant \frac{|X \cap A'|}{|X \cap A|} m_d^{A'}(s) + \frac{|X \cap A''|}{|X \cap A|} m_d^{A''}(s). \tag{A2}$$

Indeed, if $\{A'_i\}_i$ and $\{A''_i\}_i$ are coverings of $X \cap A'$ and $X \cap A''$ respectively, then the union $\{A_i\}_i$ of both families is a covering of $X \cap A$. Hence there follows

$$\begin{aligned} m_d^A(s) &= \frac{1}{|X \cap A|} \min \sum_i \left(\frac{\delta(A_i)}{s}\right)^d \\ &\leq \frac{1}{|X \cap A|} \left[\sum_i \left(\frac{\delta(A'_i)}{s}\right)^d + \sum_i \left(\frac{\delta(A''_i)}{s}\right)^d\right] \\ &= \frac{|X \cap A'|}{|X \cap A|}\left[\frac{1}{|X \cap A'|}\sum_i \left(\frac{\delta(A'_i)}{s}\right)^d\right] + \frac{|X \cap A''|}{|X \cap A|}\left[\frac{1}{|X \cap A''|}\sum_i \left(\frac{\delta(A''_i)}{s}\right)^d\right]. \end{aligned}$$

From the arbitrariness of the coverings $\{A'_i\}_i$ and $\{A''_i\}_i$, relation (A2) now follows.

The subadditivity can be used to prove the convergence of the limit in the definition (2) of the function $m_d(X, s)$. Let $m_d(l, s)$ denote the supremum of $m_d^A(s)$ over all boxes $A = V(l)$ of diameter l. Any large box A with diameter l' much larger than l can be approximately divided into boxes A_i of diameter l. Now it follows from subadditivity that $m_d^A(s) \leq m_d(l, s)$. The arbitrary choice of the box A implies that $m_d(l', s) \leq m_d(l, s)$. The limit of a non-increasing sequence of non-negative numbers is always convergent. The expression

$$m_d(X, s) = \lim_{l \to \infty} m_d(l, s)$$

can now be considered as the definition of how the limit in expression (2) should be taken.

Finally let us show that $D(d) \leq d$. Assume that $s' < s$ and that $m_d(X, s') \neq 0$. From inequality (3) it follows that

$$\ln m_d(S, s') \leq d(\ln s - \ln s') + \ln m_d(X, s).$$

The expression can be written as

$$-\frac{\ln m_d(X, s)}{\ln s} - d \leq \frac{\ln s'}{\ln s}\left(-\frac{\ln m_d(X, s')}{\ln s'} - d\right).$$

In the limit $s \to \infty$ the RHS of the expression tends to zero. One obtains

$$D(d) = -\lim_{s \to \infty} \sup \frac{\ln m_d(X, s)}{\ln s} \leq d.$$

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